

# Constructing a Large Number of Nonisomorphic Graphs of Order $n$

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**Abstract:** We give a simple construction of a large number of nonisomorphic graphs of order  $n$ .

While it is easy to see that there are  $T_n = 2^{\binom{n}{2}}$  labelled graphs of order  $n$ , the enumeration of nonisomorphic graphs is a different question altogether. A counting polynomial for such graphs can be obtained as an application of Pólya's theorem (*cf.* Harary [2, pp. 180 - 185]). We give a simple construction that yields approximately  $\sqrt{T_n}$  nonisomorphic graphs of order  $n$ . We use the notation of [1].

## Basic construction

We begin with a basic construction of a disconnected graph  $G$  of even order  $n$ , which will be a subgraph of each of the nonisomorphic graphs. Let  $V(G) = A \cup B$  (disjoint union), where  $A = \{a_1, \dots, a_{\frac{n}{2}}\}$  and  $B = \{b_1, \dots, b_{\frac{n}{2}}\}$ . Add edges such that  $\langle \{a_1, a_2, a_3\} \rangle = K_3$ ,  $a_3, \dots, a_{\frac{n}{2}}$  is the vertex sequence of the path  $P_{\frac{n}{2}-2}$ , and  $\langle B \rangle \cong \overline{\langle A \rangle}$  under the isomorphism  $a_i \mapsto b_i$ . Note that  $\Delta(\langle A \rangle) = 3$  (with  $a_3$  the only vertex of degree 3) and  $\delta(\langle B \rangle) = \frac{n}{2} - 4$ .

## Construction of the nonisomorphic graphs

Let  $A_1 = \{a_8, \dots, a_{\frac{n}{2}}\}$  and  $B_1 = \{b_8, \dots, b_{\frac{n}{2}}\}$ . There are  $p = \left(\frac{n}{2} - 7\right)^2$  possible edges between  $A_1$  and  $B_1$ . For each of the  $2^p$  subsets  $S_1, \dots, S_{2^p}$  of these edges, let  $G_i$  be the graph obtained from  $G$  by adding the edges in  $S_i$ . We prove that no two of these graphs are isomorphic. To simplify the notation we use the same labels (that of  $G$ ) for the vertices of all the graphs constructed; confusion is unlikely.

**Theorem 1** For all pairs  $i, j \in \{1, \dots, 2^p\}$  with  $i \neq j$ ,  $G_i \not\cong G_j$ .

*Proof.* Suppose to the contrary that  $f : G_i \rightarrow G_j$  is an isomorphism. By construction,  $\Delta_{G_k}(\langle A \rangle) \leq \frac{n}{2} - 5$  and  $\delta_{G_k}(\langle B \rangle) \geq \frac{n}{2} - 4$  for any  $k \in \{1, \dots, 2^p\}$ . Thus for any  $a \in A$  and  $b \in B$ ,  $f(a) \in A$  and  $f(b) \in B$ . Also,  $a_3$  is the

only vertex of degree three which lies on a triangle contained in  $\langle A \rangle$ ; hence  $f(a_3) = a_3$ . Now suppose  $f(a_k) = a_l$ , where  $k, l \geq 4$ . The length of a shortest  $a_3 - a_k$  path in  $G_i$  that consists entirely of vertices of degree at most  $\frac{n}{2} - 5$  is  $k - 3$ . Hence the length of a similar  $a_3 - a_l$  path in  $G_j$  is also  $k - 3$ . This is only possible if  $k = l$ . It thus follows that  $f(a_k) = a_k$  for all  $k \geq 3$ .

A similar argument shows that  $f(b_k) = b_k$  for all  $k \geq 3$ . But then  $a_k b_l \in S_i$  if and only if  $a_k b_l \in S_j$ , a contradiction since  $i \neq j$ . ■

We have thus constructed  $2^p \approx 2^{n^2/4} \approx \sqrt{T_n}$  nonisomorphic graphs of order  $n$ .

## References

- [1] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, Chapman & Hall, London, 1996.
- [2] F. Harary, *Graph Theory*, Addison-Wesley, Reading, Massachusetts, 1969.